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COUNTERFACTUALITY AND THE PHYSICAL REALITY: LESSONS FROM QUANTUM MECHANICS [1]

Contrafactualidade e Realidade Física Lições da Mecânica Quântica

Nicholas Kluge Corrêa[2]

ABSTRACT

In this essay, I'll present an example of counterfactual physical phenomena. More precisely, counterfactual definiteness and interaction-free measurements in quantum mechanics. The example used will be the Elitzur-Vaidman bomb testing experiment, which shows how we can probe the properties of objects, even when they have not been measured, i. e., counterfactual measurements. This type of physical phenomenon seems to operate by the same laws of causality as counterfactual reasoning in decision-making. After all, how can something that doesn't happen affect the real world? I suggest that some mathematical tools used in quantum mechanics may be of interest to those who wish to better model counterfactual possibilities in the decision process or philosophers interested in better understanding the metaphysical implications of quantum mechanics.

Keywords: counterfactual definiteness. interaction-free measurements. counterfactuality. metaphysics. quantum mechanics.

RESUMO

Neste ensaio, eu vou apresentar um exemplo de fenômeno físico contrafactual. Mais precisamente, definição contrafactual e medições livres de interação em mecânica quântica. O exemplo usado será o experimento de teste de bombas Elitzur-Vaidman, que mostra como podemos sondar as propriedades dos objetos, mesmo quando não foram medidos, por exemplo, medições contrafactuais. Esse tipo de fenômeno físico parece operar pelas mesmas leis de causalidade que o raciocínio contrafactual na tomada de decisões. Afinal, como algo que não acontece pode afetar o mundo real? Sugiro que algumas ferramentas matemáticas usadas na mecânica quântica podem ser de interesse para aqueles que desejam modelar melhor as possibilidades contrafactuais no processo de decisão ou filósofos interessados em entender melhor as implicações metafísicas da mecânica quântica.

Palavras-chave: definições contrafactuais. medições livres de interações. contrafactualidade. metafísica. mecânica quântica.

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2 Programa de Pós-graduação em Filosofia, Pontifícia Universidade Católica do Rio Grande do Sul (PUC-RS). E-mail: nicholas.correa@acad.pucrs.br. ORCID: http://orcid.org/0000-0002-5633-6094.

Affecting without Existing?

> *"We can imagine the impossible, provided we do not imagine it in perfect detail and all at once."*
> David Kellogg Lewis, On the Plurality of Worlds.

Counterfactual thinking is a form of reasoning that involves the ability of the human mind to create contrary alternatives to what happened. In this way, events that do not occur, and thereby have no causal effect on the environment, can affect our experiences and the way human agents influence the environment. This form of reasoning evaluates what might/may have happened differently ("If it were this A, then it would be this B"), is a strange form of decision making that, in one way or another, occurs in human mental processes. But how can something that didn't happen affect the environment? Can human imagination be the only way that possible consequences causally affect the world?

Counterfactuals have been of interest to philosophers since the time of Leibniz in the 17th century, where the philosopher proposed the possibility of an infinite number of alternative realities (GRIFFIN, 2012), something similar to Everett's (1957) many-worlds interpretation of quantum mechanics. David Hume, in his definition of causality, offered the following clarification:

> We may define a cause to be an object followed by another, and where all the objects, similar to the first, are followed by objects similar to the second. Or, in other words, where, if the first object had not been, the second never had existed (HUME, 1999, p. 146).

But what if the first event had been different? How would the second be? On Hume's analysis, the answer may be simple: the second simply doesn't happen cause the first also didn't happen! So should we define human imagination and counterfactual reasoning as "magic" and call it a "*quod erum demonstratum*"? Are there any physical phenomena that have this same oddness? In addition to the human mind, one of the only events capable of using counterfactual possibilities to alter a real state in the environment occurs in the phenomenon of counterfactual definiteness (CDF) and interaction-free measurements. In such phenomena, events that do not occur can have a causal, measurable, effect on the physical world (ELITZUR; VAIDMAN, 1993).

In this short essay, I'll present an example of CFD, showing how we can probe the properties of objects, even when they have not been measured (counterfactually). CFD in quantum systems is one of the only physical phenomena that operate by the same laws of causality as counterfactual reasoning. Perhaps its mathematical formalism may be of interest to those who wish to better model counterfactual possibilities in the decision process, or philosophers interested in better understanding the metaphysical implications of quantum mechanics.

Violation of Causality or Breaking Temporal Asymmetry?

In quantum mechanics, CFD and interaction-free measurements happen when the possibility of an event is "used" to make a measurement. But since this event didn't happen, the object of interest, what we are trying to measure, had no causal interaction with our “measuring apparatus” (GRIFFITHS, 2012). One of the most famous examples of CFD would be the Elitzur-Vaidman Bomb-testing experiment. In this scenario we'll use properties that elementary particles (e. g., photons) have, such as non-locality and superposition, to verify characteristics of objects without interacting with them (ELITZUR; VAIDMAN, 1993).

And is worth mentioning that this sort of measurement or more than mere thought experiments, Kwiat et al (1994) proved that interaction-free measurements are indeed possible, and several other researchers have already confirmed Elitzur and Vaidman's proposal experimentally (WHITE, 1998; PUTMAN, 2009; CARSTEN et al., 2016).

A possible interpretation of this phenomenon was given by Aharonov and Vaidman (1990), in an attempt to rescue the causal influence in interaction-free measurements, but refuting the temporal asymmetry. The authors propose that the quantum state should be represented by two different unitary evolutions, one that travels towards the future and the evolution of the system and the other that evolves to the past, from the next time the system would be measured. This second state would be governed by what would occur in the future, and not in the past. Such a proposal is consistent with relativistic interpretations of quantum mechanics and doesn't affect the rest of the unitary evolution and its results since its laws are symmetrical in time. However, it suggests that future events could influence the present. So we should ask, what are we willing to give up? Our notion of causality? Or the asymmetry of time?

Superposition and Unitary Evolution

Quantum superposition can be considered as one of the main characteristics of quantum mechanics, being a definitive property that separates classical and quantum theory. Any two quantum states, or even more, can be superposed to form another valid quantum state; and inversely, a quantum state can be represented as a sum of two or more other superposed states (DIRAC, 1930, p. 12). This property of quantum states refers to the linearity of Schrödinger's equation and its solutions, so any linear combination of solutions (superposed quantum states) will also be a valid solution/quantum state (SCHRÖDINGER, 1926).

To deal with the concept of superposed quantum states we use mathematical tools like Hilbert Spaces and linear algebra, which allows us to manipulate vectors, which in quantum mechanics represent probability amplitudes. The state-space of all possible probability amplitudes (vectors) represents the set of all possible measurement results (DIRAC, 1930, p. 15). In the context of quantum mechanics, the superposition of quantum states is not like any other principle of superposition in physics, like the superposition of classical waves. We are not talking about "real" waves, but something that classical everyday language lacks words to describe: the probability amplitude of a quantum state represents the possible places that a given particle may be, which is a particle, not a wave, but a particle with a probability amplitude intrinsic to its existence and locality. Get it? In truth no one does, and analogies made to soften the violent blow of quantum mechanics can often be misleading. To truly understand the nature of what is happening, analogies like "is a wave and a particle at the same time" will not help us.

The only language that will not fool us is mathematics, and as Richard Feynman would say, Shut up and calculate. So let's calculate our way into a unitary evolution of a quantum state.

Each quantum state can be represented by a family of vectors in Hilbert space. A Hilbert space is a generalization of Euclidean space, where we aren't restricted to a finite number of dimensions (orthogonal directions). A state is denoted by $|\rangle$, where multiple vectors to a given state differ only in phase. Thus, if $|A\rangle$ and $|B\rangle$ are two distinct physical states, the linearity of Schrödinger's equation dictates that the superposition of both states, $|\psi\rangle$, also represents the entire system (DIRAC, 1930, p. 14-16):

$$|\psi\rangle= x|A\rangle+ y|B\rangle$$

An elementary particle, like a photon, can be in two or more states. But for the sake of simplicity let's assume just two states, A and B. The photon can occupy both states in a different number of ways, with different probability amplitudes of $|A\rangle + |B\rangle$. The different phase representations for these states are given by x and y, which in turn are expressed by complex numbers, a+bi. Where a and b are real numbers and i is the imaginary unit with the property:

$$i2= -1 \rightarrow i= -1$$

where at least x or y must be not 0. It is the use of complex numbers as probability amplitudes that helps the mathematical modeling of superposed quantum states, something that in standard probability theory, where only real values can be probabilities, does not occur (HERNANDEZ, 2019). Schrödinger's equation is a mathematical description of how a quantum state changes with time. This evolution over time is also called unitary evolution. The quantum state, or wave function, is generally represented by the Greek letter ψ, where ψ it is the sum of all probability amplitudes, different phase representations, and possible states for the system, $xA\rangle+ yB\rangle+ w|C\rangle ...+ z|D\rangle$. Being thus the superposition of all possible states of the vector $|\psi\rangle$ (DIRAC, 1930, p. 10-14).

Mach-Zehnder Interferometer

We can use as an example the following experiment to better understand the unitary evolution: a photon emitter, a very weak light source that produces a single photon at a time, emitting it towards a half silver mirror, which is an optical device that divides a beam of light in two. Since the photon is an elementary particle and cannot be divided (there is no such thing as "half" a photon), it has a probability of 12 to pass through the half silver mirror and go through the "lower path", or 12 of probability to be reflected at an angle of 90º degrees and go through the "upper path" of the experiment (Figure 1).

Quantum mechanics dictates that, instead of the photon passing through or being reflected by the half silver mirror, the photon is placed in a superposed state, where it simultaneously passes through and is reflected by the half silver mirror (DIMITROVA; WEIS, 2008). Thus if the initial vector state of the emitted photon was $|A\rangle$, its unitary evolution would become the overlapping state $|B\rangle+i|C\rangle$, where $|B\rangle$ would represent the photon that passes through the half silver mirror, and $|C\rangle$ the state in which the photon is reflected. The factor i represents a change of phase respective to π since photons change phase by 180° when they reflect from the surface of a medium with a refractive index higher than that of the medium in which they are traveling (PEREIRA et al., 2009).

Now both overlapping states, $|B\rangle$ and $|C\rangle$, are directed to two common mirrors, a mirror located on each possible path, upper and lower. They are positioned to redirect the photon so that the two paths cross on another half silver mirror. This half silver mirror is positioned between the intersection of the lower and the upper path after being reflected by the common mirrors. Aligned with this half silver mirror are positioned two photon detectors, and the photon can be picked up in detector 1 or 2, but never in both. The proposed experiment, Mach-Zehnder Interferometer, can be viewed below.

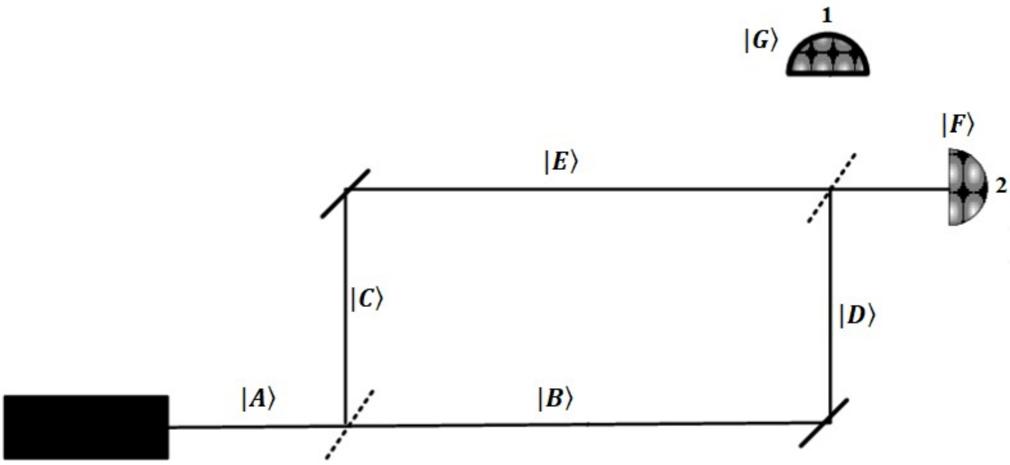

Figure1: Mach-Zehnder Interferometer

The evolution of the emitted photon, state |A⟩, would be given by the following process:

i. |A⟩ |A⟩ after crossing/reflecting on the half silver mirror becomes |B⟩ +i|C⟩;

ii. Both |B⟩ and |C⟩ are reflected by the mirrors, and evolve from |B⟩→ i|D⟩ and |C⟩→ i|E⟩;

iii. Thus, |B⟩+i|C⟩ →i|D⟩+i(i|E⟩), and since i2= -1, then i|D⟩-|E⟩. Now both photons reflected by the mirrors are again passing through a half silver mirror, where;

iv. The state |D⟩ evolves to a combination of |G⟩+i|F⟩, where |G⟩ is the transpassed state and |F⟩ the reflected one. And the same occurs with |E⟩, but now |F⟩ is the transpassed state and |G⟩ the reflected state;

v. Thus, |D⟩ → |G⟩+i|F⟩ and |F⟩ → |F⟩+i|G⟩;

vi. The i|D⟩-|E⟩ state, after passing through the second half silver mirror, is represented as follows;

i|D⟩-|E⟩ iG⟩+iF⟩-(|F⟩+i|G⟩) = i|G⟩-|F⟩-F⟩- iG⟩= -2|F⟩

Thus, there is no possibility of detector 1, |G⟩, being activated, and the superposed photon collapses into a single possibility, |F⟩. Such event occurs because the simultaneity of the photon in different states leads to a phenomenon called interference, that is, the superposed states of the photon interfere with themselves, remaining still a unique entity, its existence is affected by its counterfactual co-existence, the other way that the photon could have traveled (DIMITROVA; WEIS, 2008).

It is reduced from $xA\rangle + yB\rangle$ to $|A\rangle$ or $|B\rangle$. This phenomenon is called state vector reduction, or wave function collapse. The formalization of this phenomenon is acquired with the square module of the wave function, $|\psi|2$, which is a real number, interpreted as the probability of finding a particle in a given place at a given time. However, this form of probabilistic measurement requires that the results be normalized, that is, that the x and y values be multiplied by a complex factor so that the sum of the probabilities of the possible states is unitary ($|\psi|2 = 1$) . The product of a complex number with its conjugate is always a real number, and this phenomenon, the collapse of the wave function, transports quantum theory to classic, probability amplitudes to classic probabilities (SCHLOSSHAUER, 2005). The "why" this occurs is a reason for ongoing debate among physicists and philosophers and remains an open problem (The Measurement Problem).

The Elitzur-Vaidman bomb testing experiment

By modifying the experiment reported above a little, we now place an obstruction in the lower path after the reflecting mirror so that the obstacle absorbs the state/photon $|D\rangle$ (Figure 2). In this way, the interference previously exemplified would be canceled, therefore if the photon is not absorbed by the obstruction there is now the possibility that a photon is detected by sensor 1 as well as by sensor 2. If the photon is absorbed by the obstacle both states $|G\rangle$ and $|F\rangle$ will not be reached due to the collapse of the wave function, but if the opposite is the case;

i. the state $|A\rangle$ evolves to $i|C\rangle$;
ii. $i|C\rangle \rightarrow iC\rangle \rightarrow -|E\rangle$;
iii. $-|E\rangle$ evolves to $-F\rangle - iG\rangle$;
iv. And finally, the square module of the superposed state $-F\rangle - iG\rangle = -12 : -i2$, thus, the ratio between both states is 1:1, and there is now the equal probability of sensor 1 or 2 detecting a photon.

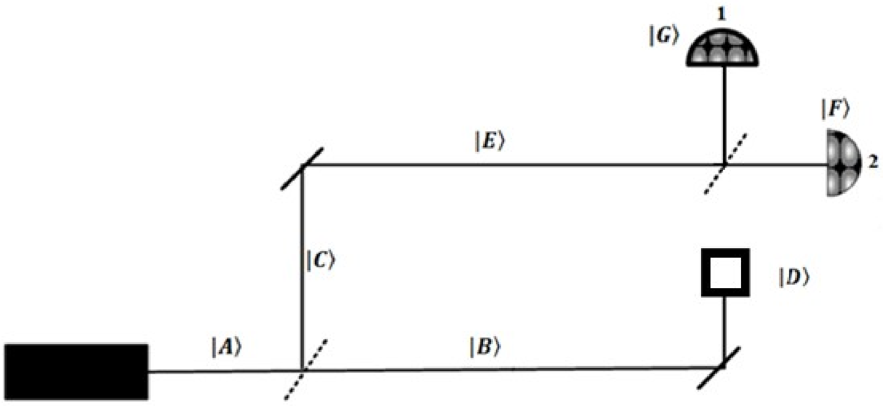

Figure 2: Elitzur-Vaidman Bomb Experiment

Now we visualize the following task: We have a collection of extremely sensitive bombs, so sensitive that if a single photon would interact with their "optical trigger" it would cause the detonation of the device. However, some of the bombs are duds, not detonating at all, allowing the photon to cross the optical trigger and follow its path. Would it be possible to separate the real bombs from the fake ones without detonating all the real devices? The answer is: yes. The Elitzur-Vaidman Bomb is an experiment that uses interaction-free measurement, a type of measurement that detects the presence of an object without causally interacting with it, and in our case, checks if a bomb is functional without having to detonate it. The experiment would result in three types of results:

1. The bomb explodes. Photon was in the state $|B\rangle$;
2. The bomb does not explode. Photon evolves from the $|C\rangle$ state to the $|F\rangle$ state. It could still be a live bomb;
3. The bomb does not explode. Photon evolves from the $|C\rangle$ state to the $|G\rangle$ state. The bomb is live, and it's not a dud.

In case 2, the dud allows the photon to traverse the optical trigger without impediments, which leads to a situation identical to the one exemplified in Figure 1, where only detector 2 is activated. However, in case 3, the bomb itself serves as an observation, causing the collapse of the wave function, exactly as in the case where we have an object obstructing the lower path of the photon. Thus, the interference is canceled, allowing the photon to evolve from the $|E\rangle$ state to the $|G\rangle$ state. Along with the test, 12 of the real bombs will be detonated and only 12 of the times that the bomb is not detonated, the unitary evolution will result in state $|G\rangle$. This way we can recover only 14 of the tested bombs of a certain set. Repeating this process, with more and more sets of bombs from our stock would result in the following sum;

$$14+116+164+1256\ldots=13$$

Therefore we could recover 13 of the real bombs if we had an infinite stock of them. With a change in the arrangement of the experiment, rearranging the angles and distributions of mirrors, this number can be reduced to 12 (VAIDMAN, 2003). Kwiat et al (1995) demonstrated that the percentage of bombs recovered, i.e., the accuracy of an interaction-free measurement, can arbitrarily approach 100%. Using another quantum phenomenon, the quantum Zeno effect, where the unitary evolution of a quantum system can be interrupted, as long as the system is measured with enough frequency, in principle all bombs could be recovered (SUDARSHAN; MISRA, 1977).

One of the modern applications for this type of model is in the counterfactual quantum calculation, introduced by Mitchison and Jozsa (2001), which consists of a method of inferring the result of a calculation without actually performing quantum computing. This concept has been explored in the area of communication (LIU et al., 2012). Currently, the efficiency limits of counterfactual computing are 85% accuracy. That is, with a quantum computer at our disposal and an arsenal of ultra-sensitive bombs, it would be possible to savage 85% of live ones (KONG et al., 2015).

Non-localism or False-Realism?

In the phenomenon of interaction-free measurements an object is found because it could have interacted with a fundamental particle, although in fact, it did not. This idea, when applied in counterfactual computing, allows the result of a calculation to become known, even if the algorithm did not perform the given computing. If we take seriously Everett's Many Worlds interpretation the counterfactual possibilities are part of a single totality. Thus, the physical universe incorporates all realities, and all would be interconnected by counterfactually influencing each other. This idea also provides a different perspective for the phenomenon of non-locality, non-locality being the impossibility of consistent coexistence of results with alternative inputs. Thus, locality should be something that embarrasses the entire physical universe and its counterfactual neighborhood. Therefore, the so-called "non-local" phenomenon would not involve any non-locality, the problem of local realism not being a problem of locality, but of realism, and what we understand as reality.

Applications for Decision Theory

Another important idea that we can draw from the above examples is the possibility of modeling counterfactuality in decision making as superposed states, using the same mathematical formalism used to formalize CDF in quantum mechanics (linear algebra and non-kolmogorovian probability). Would it be possible to use the concept of Hilbert's spaces in the context of decisions making under uncertainty? That is, to implement an abstract space of possibility in decision theory. In quantum mechanics, a wave function ψ is a vector in a Hilbert space and represents a range of probability amplitudes. Could an n-dimensional vector code the probability distribution between possible actions and beliefs of a decision-maker?

Thus, just as quantum entanglement is a consequence of the Schrodinger equation, this concept can be useful in understanding decision making, where decision making can be understood as the superpositioning of counterfactual possibilities. While classic decision theory is based on the subjective interpretation of probability theory, allowing a dynamic update of beliefs according to the Bayesian rule, a linear algebra approach allows a dynamic that evolves along with the system over time. This interpretation can help the development of new tools and concepts for decision theory in general.

Utilizing the mathematical tools and concepts of quantum mechanics to describe counterfactual behavior in decision making should not be interpreted as evidence that quantum phenomena influence human decision making. What is implicit in the proposed suggestions is that both phenomena can be modeled by the same mathematical formalism. Currently, there are already some proposals of "Quantum Decision Making Theories" (YUKALOV; SORNETTE, 2018; AERTS et al., 2016; 2017; 2018). However, this terminology can be misleading. I believe it would be better to define such proposals, to avoid unnecessary quantum woo, as State-Dependent Decision Theory or Superposed-States Decision Theory.

The use of probabilistic models adopted by quantum physics could help us better understand a large number of paradoxes and problems faced by standard decision-making theory. Could these paradoxes and problems be an indication that the Kolmogorov theory of probability may not be sufficient to describe probabilistic phenomena involving human behavior? Even so, the question of whether Kolmogorov's formalism is sufficient to model all probabilistic phenomena, or isn't, it's a philosophical problem. Since, depending on the way we define "probability", we can get different answers. At least in the case of quantum mechanics, we already know that it is not enough, and the standard probability theory has to be extended. I suggest that similar extensions in decision-making models should be investigated more seriously, given the intuitive similarities between counterfactual physical phenomena, and counterfactual thinking.

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