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# ON THE IMPREDICATIVITY AND CIRCULARITY OF FREGE'S ANCESTRAL

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## ABSTRACT

Among Frege's major contributions to the philosophy of mathematics lies his famous definition for the Ancestral relation. Presented first in the 1879 *Begriffsschrift*, it has an important role in Frege's logicism, given that it provides the conditions for transforming the predecessor relation into a linear-ordered series. One of the objections for Frege's definition comes from its impredicativity and the circularity thus yielded. Such objection was pointed by Benno Kerry in 1887, and more recently by Ignacio Angelelli in 2012. In this article, I argue from a Fregean perspective that Frege's Ancestral is not circular, although its inevitably impredicative character.

**Keywords:** Gottlob Frege. Ancestral Relation. Impredicativity. Logicism.

## RESUMO

Entre as principais contribuições de Frege à filosofia da matemática encontra-se a sua famosa definição de Ancestral de uma Relação. Apresentada primeiro na *Begriffsschrift* de 1879, tal definição possui um papel crucial no logicismo fregeano ao estabelecer as condições para transformar a relação predecessora em uma série linearmente ordenada. Uma das objeções à definição de Frege provém de sua impredicatividade e da circularidade resultante. Tal objeção foi primeiramente apresentada por Benno Kerry em 1887, e mais recentemente por Ignacio Angelelli em 2012. Neste artigo, argumento, a partir da perspectiva fregeana, que o Ancestral não é circular, não obstante sua inevitável impredicatividade.

**Palavras Chave:** Gottlob Frege. Relação Ancestral. Impredicatividade. Logicismo.

## 1. Logicism and Frege's Ancestral

Frege's definition of the Ancestral, as stated in the third section of the *Begriffsschrift*[2], is a second-order definition for the transitive closure of any first-order relation. For any first-order relation $R$ and objects $a$ and $b$, we say that Frege's Ancestral, denoted by $R^*(a,b)$, holds if, and only if, $b$ has all the

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[2] Following the English translation in (FREGE, 1967). I reserve the German *Begriffsschrift* as denoting the 1879 book, and *concept-script* as denoting the logical notation presented in it.

hereditary properties that $a$ has and that all immediate $R$-successors of $a$ also have. Formally, this is

$$R^*(a,b) =_{\text{def}} \forall F[Her(F,R) \wedge \forall z(R(a,z) \rightarrow F(z)) \rightarrow F(b)]$$

with $Her(F,R)$ meaning "$F$ is hereditary in $R$", formally

$$Her(F,R) =_{\text{def}} \forall x \forall y (F(x) \wedge R(x,y) \rightarrow F(y))$$

That is, $F$ is a property that is passed along the objects satisfying the relation $R$. The name "Ancestral" has its origin in the fact that, if we consider $R$ as denoting the parent relation, then $R^*$ is the ordinary ancestral relation. It is also considered that Frege's definition is a logical reduction of this ordinary notion.

This was an important step for Frege's logicism since it provides the conditions to transform the predecessor relation into a linear-ordered series, being both transitive and trichotomous. The predecessor relation $P$ is defined in Frege's *Grundlagen der Arithmetik,* §76[3] as:

$$P(m,n) =_{\text{def}} \exists F \exists x [F(x) \wedge n = \# F \wedge m = \# [z : F(z) \wedge z \neq x]]$$

Where we read that $m$ precedes $n$ if, and only if, $n$ is the number of the objects satisfying a given concept and $m$ is the number of such objects, except for one of them. Here, # is a number operator, denoting the number that is associated with the concept $F$[4]. With such relation, $P^*$ denotes the series of objects that are correlated through $P$, being $P^*$ both transitive and trichotomous. First, it's provable that

$$P^*(x,y) \wedge P^*(y,z) \rightarrow P^*(x,z)$$

Which is an instance of Frege's theorem 98 of the *Begriffsschrift* about the transitivity of the Ancestral, that provides the basic connection between

[3] Following Austin's translation in (FREGE, 1953). Henceforth quoted simply as *Grundlagen*.

[4] Frege introduces it with the abstract principle known as Hume's Principle in *Grundlagen*: $\# F = \# G \equiv F \approx G$, where $\approx$ is a bijection between the $F$'s and the $G$'s.

each and every element linked through $R$. Frege also offers the following two definitions: The Weak-Ancestral of $R$

$$R^{+}(x, y) =_{\text{def}} R^{*}(x, y) \vee x = y,$$

and the Single-Valuedness of a relation

$$Fun(R) =_{\text{def}} \forall x \forall y \forall z (R(x, y) \wedge R(x, z) \rightarrow y = z).$$

Using the former, Frege defines the natural numbers in terms of following the number zero in the series generated by the predecessor relation or being equal to zero, that is,

$$N(x) =_{\text{def}} P^{+}(0, x)$$

Following the fact that the predecessor relation is functional, *i.e.* single-valued[5], it can be proved as an instance of Theorem 133 of the *Begriffsschrift*:

$$N(x) \wedge N(y) \rightarrow (P^{*}(x, y) \vee P^{*}(y, x) \vee x = y),$$

which means that, for any pair of natural numbers, trichotomy holds for them. This yields that the series of natural numbers is linear-ordered.

The Ancestral also makes it possible to derive a general principle of induction, Theorem 81 of the *Begriffsschrift*,

$$[F(x) \wedge Her(F, R) \wedge R^{*}(x, y)] \rightarrow F(y)$$

and the principle of mathematical induction in the form[6]:

$$F(0) \wedge \forall v \forall w (N(v) \wedge F(v) \wedge P(v, w) \rightarrow F(w)) \rightarrow \forall z (N(z) \rightarrow F(z))$$

As a major result from all these facts, the Peano-Dedekind axioms for arithmetic are derivable in Frege's logic, assuming, of course, that the number-operator # is well-defined and Hume's Principle as a legitimate substitute

[5] This is declared in the *Grundlagen*, §72, proposition 5.

[6] Assuming the necessary corrections pointed out by Boolos and Heck in (BOOLOS & HECK, 2011)

for Frege's Basic Law V. Frege's definition of the Ancestral, then, has a substantial role in Frege's logicism.

Philosophically, Frege's Ancestral was his first argument against the necessity of kantian intuitions for ordered series and the foundations of Arithmetic, as well for the informativity of (second-order) logic. Kant's philosophical position towards numbers and arithmetic, as exposed in the first *Critique* (KANT, 1998), takes logic as a formal discipline, a *canon* of reason which does not have any content of his own. For this reason, Kant believed that pure intuitions were necessary to preserve both the content and the *a priori* character of the arithmetical science. In this scenario, Kant's argument for the well-ordering of arithmetic is the following: Numbers are to be regarded as symbolic constructions in time, one that depends upon the counting or adding homogeneous units in each interval. Since time, being uni-dimensional and unlimited, is a linear order by default, each number constructed preserves its position in the timeline. The well-ordering of the natural numbers is then an easy corollary for Kant.

Frege's position against Kant starts by defining a way to characterize linear orders without the pure intuition of time. This involves a criticism of the contentless character of Kant's notion of formal logic and the triviality of analytic judgments. Frege takes his concept-script logic to be contentful, and the formal results regarding the Ancestral about linear-ordered series to be his first step towards this goal. He explicitly mentions this in the *Grundlagen*, in regarding the proof of theorem 133 as being not only contentful but also analytic, remarking how "From this proof, it can be seen that propositions which extend our knowledge can have analytic judgements for their content" (FREGE, 1953, §91). This is evidence that already in 1879 Frege was not only endorsing logicism but also criticizing Kant's position, something that was only made clearer in the *Grundlagen*, five years later. All this through the Ancestral definition.

But Frege's logicism, despite all its merits, did not survive Russell's Paradox, following the inconsistency of Basic Law V of the *Grundgesetze der*

*Arithmetik* system[7]. The failure of Frege's axiom V was not just an isolated problem, but something that undermined mostly every attempt into the foundations of mathematics that used an unrestricted notion of sets or classes. Russell, too, was affected by it and attempted his own amendment in the years following the discovery of the paradox. The root of the problem, something that unrestricted comprehension axioms presuppose, was that any formula could define a set or class[8]. But Russell's Paradox shows otherwise. It cannot be the case that expressions like "$x \notin x$" define a set or class. Russell (1907) called such cases as *non-predicatives*, later labeled as *impredicatives*. They are distinguished from those expressions that can define a set as *predicatives*. But what exactly prevents such impredicative expressions from defining sets was better argued later by Poincaré[9]: the presence of a vicious circle, or the attempt to define an object in terms of a domain that already contains it. Russell's Paradox is a prime example where such a vicious circle leads to paradoxical situations, and for that reason, Russell's solution was to ban such unrestricted quantification, which evolved into his Ramified Type Theory. This was done based on the *vicious circle principle*: that no object (or set) should be defined in terms of itself, or the aggregate in which it is an element.

But are all impredicative definitions harmful? The question whether impredicative definitions should be accepted was intensively debated in the first half of the twentieth century, to the point of Carnap calling it the "The *most difficult problem* confronting contemporary studies in the foundations of mathematics" (CARNAP, 1964, p.49), especially because predicative principles, such as Russell's vicious circle principle and the ramified theory of types, were successful in avoiding paradoxes, but were inadequate for dealing with some portions of mathematics, more crucially with the real numbers[10]. But in

[7] Following Ebert and Rossberg's translation in (FREGE, 2013).

[8] Frege did accept comprehension principles for concepts in 1879, and functions in 1893. However, he did it not in terms of axiom schemes, but through an undeclared rule of uniform substitution. Nonetheless, it is only with the presence of Axiom V that comprehension is problematic. In axiomatic set-theory such as ZFC, in order to avoid such problems, a limited version of comprehension is adopted as an axiom schema of separation.

[9] See, for example, chapter IV in (POINCARÉ, 1963). For a discussion on this topic, see also (FEFERMAN, 2005).

[10] Carnap's quote continued: "How can we develop logic if, on the one hand, we are to avoid the danger of the meaninglessness of impredicative definitions and, on the other hand, are to

the philosophical side of the debate, a common wisdom was formed: that impredicativity was generally acceptable under a Platonistic background[11].

Regardless if such common wisdom is faithful to the debate, we can extend it back to Frege's Ancestral. Already in 1887, Benno Kerry pointed out that Frege's definition was impredicative, given its circular nature. Since Frege's strategy was to define natural numbers in terms of the Ancestral, this definition was doomed to fail, according to Kerry. More recently, Ignacio Angelelli (2012) argued the same, adding that such circularity also undermines Frege's reduction of the very notion of a series in logical terminology. My main goal here is to defend that, from Frege's point of view, *i.e.,* his realist background, the definition of the Ancestral is not as harmful as considered by Kerry and Angelelli. As a bonus, and since Frege never answered Kerry, I'll argue that Frege could have been the first realist defense of impredicativity[12].

Towards this goal, I'll restate both Kerry's and Angelelli's arguments against Frege's definition to then recapitulate Frege's reduction and some of his philosophical justification for it. This will set the course for the second point: that the circularity is true only under a questionable assumption about quantification in Frege's concept-script and logic in general. It will help us understand why such circularity is not a problem for Frege, and perhaps why he never answered Kerry's criticisms. After comparing other famous answers to Russell's vicious circle principle with the Fregean one here proposed, I'll conclude that, at least in this scenario, Frege's Ancestral is not circular, although still impredicative.

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reconstruct satisfactorily the theory of real numbers?" (CARNAP, 1964, p.49). For instance, quantification over the reals was seriously limited by Russell's ramified theory, forcing some reinterpretation of basic expressions such as "for all real numbers." Moreover, classical mathematics is seriously limited if not by assuming some impredicative instances.

[11] This is summarized and evaluated by Parsons (2014). Some of the authors discussed will be assessed later.

[12] This, I should mention, must be read regarding only the impredicativity of the Ancestral. It is well known that Frege's logic is impredicative also in the sense of adopting full comprehension schemes for formulas, which, adding axiom V, is responsible for the inconsistency of Frege's second-order logic. But no inconsistency is known regarding the impredicativity of the Ancestral.

## 2. The impredicativity of Frege's Ancestral

Textually, Frege's definition of the Ancestral states that $x$ is the $R$-ancestor of $y$ (*i.e.*, $y$ follows $x$ in the $R$-series), just in case $y$ has all $F$-hereditary properties shared by all $R$-successors of $x$ (*i.e.*, all objects that follows $x$ in the $R$-sequence). This is a second-order definition since it quantifies over the domain of concepts, which in Frege's case, is unrestricted. In addition to the more famous debate on his theory of concepts, Benno Kerry also criticized Frege over this definition. This was done essentially in 1887 in *Über Anschauung und ihre psychische Verarbeitung* (KERRY, 1887, p.295), where he states that:

> Now, this criterion is to begin with of dubious value because there is not a catalogue of such properties, hence one is never sure that one has examined the totality of them. Moreover, there is the crucial fact that, as the author himself has proved [in a footnote Kerry cites *Begriffsschrift,* Theorem 97], of the properties that are hereditary in the $f$-series is also the following: to follow $x$ in the $f$-series. Thus, the determination of whether $y$ follows $x$ in the $f$-series, according to the definition given for this concept, depends on whether, in addition to a lot of other things on hereditary properties in general, one knows, in particular, about the hereditary property "being a descendant of $x$", that $y$ has it or not. It is clear that this circle should totally prevent from saying, in Frege's sense, that any $y$ follows $x$ in an $f$-series[13].

In justifying if $R^*(x, y)$ holds or not, it is required that, for every hereditary property $F$, one can decide whether $F$ is one of the properties that is passed along from $x$ to $y$ or not. Kerry's point is that since Frege's definition quantifies over all hereditary properties, there could be one for which this task is uncertain. The case in question, as Kerry quotes, is Frege's theorem 97:

$$\forall u \forall v \left(R^*(x,u) \wedge R(u,v) \rightarrow R^*(x,v)\right)$$ [14]

This theorem states that the property $[z: R^*(x,z)]$, which we can read as "following $x$ in the $R$-series", is hereditary in $R$. This is the same as saying that the property "being $x$ descendant" is something that every descendant of $x$ pass along the parent relation. For example, if Gottlob is Karl's descendant,

[13] The translation is Angelelli's (ANGELELLI, 2012). Where he translates "being a descendant of $x$", Kerry actually writes simply "to follow $x$", since he, as Frege, does not name such relation as Ancestral.

[14] I avoided here the *definiendum* of the Hereditary Property, showing only the *definiens*.

then Alfred, Gottlob's son, is Karl's descendant as well. The problem pointed out by Kerry is that to determine whether Alfred is Karl's descendant, we have to find all such hereditary properties and check if Alfred has them. But in this process, the above property would require to determine if Alfred is Karl's descendant. Hence the circularity.

This circularity was later reassessed by Ignacio Angelelli in *Frege's Ancestral and his Circularities* (ANGELELLI, 2012). He presented two versions, the first one being Kerry's, as stated above. The second one aims to show how Frege's intended reduction of the notion of a series failed as well. Simply put, a series is any connection between a set of elements that is at least transitive. It states that there is a "chain" between one object to another given a finite number of steps. Frege did not name such notion as the Ancestral, but such relation is equivalent: both are transitive connections, or paths, between one element (the ancestor) to another (the descendant). This is what Angelelli defines as the Ordinary Ancestral, henceforth labeled as **(OA)**.

Frege's definition was a reduction of **(OA)** into logical terminology. This is what he argues in (FREGE, 1967, p.5): "My initial step was to attempt to *reduce* the concept of ordering in a sequence to that of *logical* consequence, so as to proceed from there to the concept of number"[15]. This is also, according to Angelelli's interpretation, what Frege maintains in defining the Ancestral, in (FREGE, 1953, §79), by saying that the *definiens* "is to mean the same as" [*sei gleichbedeutend*] the *definiendum*. For that reason, Angelelli concludes that:

> It seems natural do interpret the product of the reduction as intended to replace the initial notion. In alternative terms, it seems natural to construe the Fregean Ancestral as an *analysans* that replaces, in Frege's project, the *analysandum* (the common ancestral). Such would be the analysis interpretation. (ANGELELLI, 2012, p.478)

Since Frege was also interested in developing a logical system capable of replacing the natural language, his logical definition of the Ancestral, henceforth **(FA)**, should be taken as a replacement of **(OA)**. But was Frege's intended reduction successful? According to Angelelli, no.

[15] The first emphasis is mine. The second is Frege's.

Kerry and Angelelli's versions of the circularity have the same basic principle: that **(FA)** is circular given its unrestricted quantification over properties. I focus here on Angelelli's versions, with the following fictional situation added by him: a certain Fritz is trying to convince a jury that he is Karl's descendant to inherit Karl's money. Since Fritz doesn't have the necessary documents, he is tempted to quote Frege's definition, *viz.,* that he is Karl's descendant if, and only if, he has all hereditary properties shared by Karl's descendants. The jury then asks him to check whether Frege's definition could help him with two specific properties: "being the Fregean-descendant of Karl" and "being the ordinary-descendant of Karl." Recall that **(FA)** is $R^*(x, y)$, and consider now $R^o(x, y)$ as denoting that $x$ is the ordinary ancestor **(OA)** of $y$[16]. The properties questioned by the jury are $[z: R^*(k, z)]$ and $[z: R^o(k, z)]$ respectively, where $k$ is a name for Karl. Assuming that Fritz wants to check such properties, the first one generates Kerry's circularity. The second exemplifies Angelelli's argument against Frege's reduction of the Ancestral, what he calls the analysis interpretation. More precisely, and assuming that $k$ and $f$ are names for Karl and Fritz, the first is the following:

**Kerry's Circularity:**

(a) In order to prove that $R^*(k, f)$ holds, one has to show all $F$ properties which are hereditary in $R$ such that, if $\forall z(R(k, z) \rightarrow F(z))$ then $F(f)$;

(b) $[z: R^*(k, z)]$ is such property;

(c) Then, one has to show that $f$ has the property $[z: R^*(k, z)]$, *i.e.*, that $R^*(k, f)$ holds;

(d) This is circular, hence, $R^*(k, f)$ cannot hold.

The second circularity is an argument against the Analysis Interpretation. It goes as follows:

[16] This simply means that $R^o(x, y)$ should be read intuitively as the existence of a path from $x$ to $y$.

**Angelelli's Circularity**

(a) Assume that **(FA)** is a reduction of **(OA)**;

(b) From (a), to show that $R^o(k,f)$ holds, one has to show that $R^*(k,f)$ holds;

(c) From **(FA)**, to prove that $R^*(k,f)$, one has to show all $F$ properties which are hereditary in $R$ such that, if $\forall z(R(k,z) \rightarrow F(z))$, then $F(f)$;

(d) $[z: R^o(k,z)]$ is such property;

(e) Hence, one has to prove that $f$ has the property $[z: R^o(k,z)]$, that is, prove that $R^o(k,f)$ holds;

(f) This is circular. Therefore, **(FA)** is not a reduction of **(OA)**.

Both circularities can be generalized for any $x$ and $y$. The first was already discussed above. The second one goes between (b) and (e). It simply states that if one wants to prove that someone is his ordinary ancestor in terms of Frege's definition, at some point, he would have to prove that he is the ordinary ancestor of that person. Hence, the proof would require the conclusion as one of its premises. The consequence of such circularity is that to **(FA)** successfully reduce **(OA)**, **(OA)** should be stated in terms of **(FA)** alone. Still, no such reduction is possible since every time we want to show that **(OA)** holds in terms of **(FA)**, we are obligated to prove **(OA)** in advance.

Angelelli's conclusion, then, is that at best **(FA)** is an enrichment and a generalization of **(OA)**, not an analysis or reduction. In his words:

> The enrichment occurs through the "discovery" of the property of being hereditary that many properties have [and] includes the focusing on, and helps towards the demonstration of the formal properties of the ordinary ancestral, *e.g.*, transitivity, which as Frege points out is what leads to the logical understanding of arithmetical induction. The generalization is accomplished in that the ancestral's underlying relation as such is conceived in a most abstract fashion. (ANGELELLI, 2012, pp.498-499)

Since **(FA)** quantifies unrestrictedly over the domain of properties, the ordinary ancestral still appears in its scope. For that reason, **(FA)** cannot accurately substitute **(OA)**. The alleged enrichment of **(FA)** has already been discussed above. It is not only transitive, but given the suitable conditions **(FA)** is also trichotomous and fundamental for the natural number series and the principle of mathematical induction. The generalization is due solely to the fact that the Ancestral is actually not a first-order relation *per se*, but a second-order property that can be applied to any first-order relation. Either way, Frege's definition is impredicative, as the first circularity clearly shows.

But yields the impredicative nature of **(FA)** a circular definition? Is it as harmful as Kerry and Angelelli supposed? At least in Fregean standards, it isn't. Both arguments have, at least, one problematic premise. Added with Frege's philosophical motivations, this might help to understand and avoid both circularities.

## 3. Frege's Reduction: Logicism and Realism

As mentioned earlier, logicism was Frege's primary motivation from the beginning, one that could very well be stated as the attempt of freeing arithmetic from intuitions. This was done mainly in two fronts: first, in showing that arithmetical concepts are reducible into logical terminology and second, in showing that arithmetical modes of inference are reducible to logical modes of inference. Angelelli is well aware of these motivations. He recalls two crucial points: (1) that for Frege any consideration about features of particular cases of **(FA)** are not essential and (2) that the proof of an instance of $R^{*}(x, y)$ should not be an enumeration of each point of the chain that starts from $x$ and leads to $y$. These are essential points that Angelelli rightfully mention, but failed to link to the circularity problem. The first thing, argued in the *Begriffsschrift*, is the generality of **(FA)**:

> The propositions about sequences developed in what follows far surpass in generality all similar propositions which can be derived from any intuition of sequences. Therefore, if one wishes to consider it more appropriate to take as a basis an intuitive idea of sequences, then he must not forget that the propositions so obtained, which might have somewhat the same wording as the ones given here, would not state nearly so much as these because they would have validity only in the domain of the particular intuition upon which they were founded. (FREGE, 1967, §23)

It follows that Frege's definition is one that generalizes the basic notion of a (transitive) series. Thus, examples as the ordinary ancestral relation holding between human beings, the relation between a number following another in the natural number series, or any other partially ordered series are instances of Frege's definition. Particular cases of **(FA)** can be about objects that are only apprehended from some intuition or none at all, but **(FA)** does not have such restrictions. This is confirmed by the fact that theorems regarding it are proved formally, or as Frege would put it, from "pure thought" alone (FREGE, 1967, §23). The generality of the definition is again mentioned in (FREGE, 1953, §80): "since the relation $R$ has been left indefinite, the series is not necessarily to be conceived in the form of a spatial and temporal arrangement, although these cases are not excluded." Following this passage, Frege starts arguing against what we might consider a step-by-step proof procedure for the Ancestral. This is: "if starting from $x$ we transfer our attention continually from one object to another to which it stands in the relation $R$, and if by this procedure we can finally reach $y$, then we say that $y$ follows in the $R$-series after $x$" (*ibidem*)[17]. From this procedure, if Fritz wants to prove that he is Karl's descendant, he just has to prove that he is the son of a son of ... a son of Karl, something like:

$$\exists x_1, \ldots, \exists x_n \left( P\left(k, x_1\right) \wedge, \ldots, \wedge P\left(x_n, f\right) \right)$$

where $P$ is the usual parent relation. A step-by-step proof would require that each link of the chain between Karl and Fritz be proved to exist. Frege does not rely on this kind of proof. He continues by saying that "this describes a way of *discovering* that $y$ follows, it does not define what is *meant* by $y$'s following" (*ibidem*)[18]. Surely, a discovery would be enough to *show* that Fritz follows Karl in the Ancestral relation, but is this enough for a proper logical analysis? Fritz ancestry does not depend upon his proof in front of the jury, it depends on some conditions that obtain by virtue of him *being Karl's descendant*, which, in Frege's analysis, depends on him having some required pro-

[17] Frege uses an uppercase $\Phi$ to express the relation that here, and henceforth, is denoted by the usual roman uppercase $R$.

[18] The emphasis is mine.

perties that are a consequence of the fact that such relation between Karl and Fritz obtains. Thus, for a proper logical analysis, we should not take the discovery as relevant, neither should we define a concept or relation that depends on it. As Frege argues:

> Whether $y$ follows in the $R$-series after $x$ has in general absolutely nothing to do with our attention and the circumstances in which we transfer it; on the contrary, it is a question of fact, just as much as it is a fact that a green leaf reflects light rays of certain wave-lengths whether or not these fall into my eye and give rise to a sensation, and a fact that a grain of salt is soluble in water whether or not I drop it into water and observe the result, and a further fact that it remains still soluble even when it is utterly impossible for me to make any experiment with it. (*ibidem*).

This relates to Frege's realism: that there are facts independent of our way of apprehending them. This is the case for arithmetic, since numbers are, as he regards them, self-subsisting objects. Likewise, arithmetical propositions are true independently of our way of regarding them.

This realist approach towards logic and arithmetic also assumes that properties (concepts) are objective and independent entities. They are things that we *grasp*, not things that we construct:

> [...] a concept is something objective: we do not form it, nor does it form itself in us, but we seek to grasp it, and in the end we hope to have grasped it, though we may mistakenly have been looking for something where there was nothing. (FREGE, 1984, P.133).

The same goes to definitions, as they do not have the power to create anything, only to grasp something:

> Just as the geographer does not create a sea when he draws borderlines and say: the part of the water surface bordered by these lines I will call Yellow Sea, so too the mathematician cannot properly create anything by his definitions (FREGE, 2013, p.XIII)

Fregean definitions are non-creative: they are not responsible for putting something into existence but instead are useful conventions used for inferential purposes. A Fregean definition *grasps* a concept; it does not create it. Thus, in Frege's realism, there is no problem in defining something in terms of the totality in which is itself a member, given that the object defined - in the case of the Ancestral, the second-order concept in which binary relations fall – exists independently of the definition.

> on the one hand, we can ask by what path a proposition has been gradually established; or, on the other hand, in what way it is finally most firmly establishable. Perhaps the former question must be answered differently for different people. The latter [question] is more definite, and its answer is connected with the inner nature of the proposition under consideration. (FREGE, 1967, p.103)

The realist stance also relates to Frege's anti-psychologism: roughly that we cannot take the mental representation of an object as the object itself. At least as far as arithmetical propositions are concerned, the genesis of our mental representations about some fact has nothing to do with the justification for its truth. This is argued in the very beginning of the *Begriffsschrift*:

> He then concludes that it's "[...] not the psychological mode of origin, but the most perfect method of proof underlies the classification" (*ibidem*). The point was also made in the introduction of *Grundlagen* as a *dictum*:
>
> > Never let us take a description of the origin of an idea for a definition, or an account of the mental and physical conditions on which we become conscious of a proposition for a proof of it. [...] a proposition no more ceases to be true when I cease to think of it than the sun ceases to exist when I shut my eyes. (FREGE, 1953, p.xviii)

All this reminds us of how Frege distinguished between the discovery that a proposition is true from the justification for it being true.

Returning to Fritz's case, one should expect to prove his heritage by showing each step of the chain between him and Karl. But Frege's realism towards truth assumes that this fact, whether Fritz is or isn't Karl's descendant, does not change according to his defense for it. He is Karl's descendant or not, regardless if the jury is convinced with his speech. Frege himself is saying that:

> What I have provided is a criterion which decides in every case the question Does it follow after?, wherever it can be put; and however much in particular cases we may be prevented by extraneous difficulties from actually reaching a decision, that is irrelevant to the fact itself. (FREGE, 1953, §80)

According to Frege's philosophical recommendations, $R^{*}(x, y)$ cannot rely upon a step-by-step proof, the type of which shows each link of the connection between $x$ and $y$. Theorem 98, about the transitivity of the Ancestral, could be equally proved in this intuitive way. If there is a path between $x$ and $y$, and another from $y$ to $z$, we could then prove that there is another from $x$ to $z$, by simply starting from $x$, reaching $y$ and moving along until we finally reach $z$. But should the proof depend on such a procedure? Certainly not:

> We have no need always to run through all the members of a series intervening between the first member and some given object, in order to ascertain that the latter does follow after the former. Given, for example, that in the $R$-series $b$ follows after $a$ and $c$ after $b$, then we can deduce from our definition that $c$ follows after $a$, without even knowing the intervening members of the series. (*ibidem*).

As we saw, Frege's intended to offer a definition that makes possible to prove important facts about the Ancestral in a purely logical way, with maximal generality and without any need for intuitions in its proofs. This is **(FA)**. **(OA)**, on the other hand, is the intuitive notion that Frege wants to avoid, one that requires a step-by-step proof relying on particular cases of application. Thus, and Angelelli is right about this, **(FA)** is a reduction of **(OA)**, a generalization and certainly an enrichment, one that seeks to provide for the common notion a formal and precise formulation that is purely logical. But Angelelli's and Kerry's argument for the circularity made use of an assumption that is very much akin to what Frege is trying to rule out with **(FA)**.

## 4. The circularity revisited

Why is Fritz incapable of proving his desired heritage of Karl's by simply quoting **(FA)**? Because **(FA)** doesn't work that way. By asking Fritz to prove whether he has the property "being Karl's descendant", the jury is asking for the exact step-by-step proof that Frege wanted to shun. But instead of asking proof for each link of the ancestor to the descendant, they ask for each hereditary property, those that are passed along from one to the other, to be proved in advance.

This is what happens with both arguments for the circularity of **(FA)**. The assumption is that to verify the truth of a quantified sentence like $\forall x F(x)$, one has to show first that $F$ holds for every object in the domain. Thus, a universally quantified sentence would require a justification for each of its instances to be true. Even more, it would require that the desired sentence should be accepted as true *only* if, in principle, we verify each instance in advance. But, of course, this would get us into trouble if the quantified domain is not finite, something that is not clear in the case of **(FA)**. In this scenario, sentences like "every natural number is either even or odd" would require more than is humanly possible to verify to be true. This is not what quantification is about. However, this is still presumed if such circularity is to be derived from **(FA)**: only in this scenario, the question whether $b$ has the property $[x : R^*(a, x)]$ can be raised *prior* the justification for $R^*(a, b)$.

This is what at least one premise of each argument presupposes to be valid: premises (a) and (c) from the first and second arguments, respectively. They assert that to prove $R^*(a, b)$, one has to show all $F$ properties which are hereditary in $R$ such that, if $\forall z(R(a, z) \rightarrow F(z))$, then $F(b)$. This is precisely the problem that Frege had answered in the *Grundlagen*. One needs no verification of each step of the sequence to prove that $R^*(a, b)$. Analogously, one needs no checking of each hereditary property to prove that $R^*(a, b)$. In this case, both Kerry and Angelelli seem to be taking a verification for logical justification.

First, we have a step-by-step proof of the Ancestral relation that Frege ruled out, *i.e.* the checking of each link of the series in order to verify the connection between the ancestor and the descendant. This is an epistemic verification, *viz.* a discovery. In the same way, Angelelli and Kerry seem to be assuming an epistemic verification for the application of **(FA)**: the checking of each hereditary property that the ancestor passes through $R$ and the subsequent checking of the same properties in the descendant. It's safe to say that Frege would rule that out too.

There seems to be a confusion in both circularity arguments regarding the role of the universal quantifier and its logical justification. Frege never answered Kerry about the circularity problem, but Russell had in the appendix dedicated to Frege's philosophy in his *Principles of Mathematics.* About Kerry's argument, Russell states:

> This argument, to my mind, radically misconceives the nature of deduction. In deduction a proposition is proved to hold concerning *every* member of a class, and may then be asserted of a particular member: but the proposition concerning *every* does not necessarily result from enumeration of the entries in a catalogue. (RUSSELL, 1996, p.522)

The same can be applied to Angelelli's argument. He even mention Russell's passage but doesn't discuss the response in his paper. He also mention Carnap's response, but also does not consider it. Carnap's point is in favor of Russell and Frege:

> [...] in order to demonstrate the truth of a universal sentence, it is not necessary to prove the sentences which result from it by the substitution of constants; rather, the truth of the universal sentence is established by a proof of that sentence itself. The demonstration of all individual cases is impossible from the start, because of their infinite number, and if such a test were necessary, all universal sentences and all indefinite predicates (not only the impredicative ones) would be irresoluble and therefore (by that argument) meaningless. (CARNAP, 1937, p.163)

Then, not just this proof procedure is untenable but is impossible in cases where the domain in question is infinite since such checking procedures are finite operations.

Even though Angelelli is aware and correct about Frege's intentions, I take that these points are sufficient to disqualify both circularities under Frege's background, or at least to raise significant issues on it. In Angelelli's version, in contrast to Kerry's, the focus is on the proof that $[x: R^o(k,x)]$ is a hereditary property in $R$ (premise (d)). But as I see it, this can be answered in the same way. The question of whether Fritz satisfies or not property $[x: R^o(k,x)]$ does not depend on any checking. Either $R^o(k,f)$ is the case or it isn't regardless of Fritz's arguments. In the fictional story, it's pretty clear that Fritz goes to the court to show his heritage, not to determine the truth of it. Frege's Ancestral is insufficient for the task only because it is not a definition designed for such task in the first place. The proper question is this: if Fritz is Karl's descendant, does **(FA)** holds for them? I see no reason to think that it doesn't[19].

[19] The other way to put it is by asking whether Frege's ancestral is extensionally correct in capturing the cases where there exists a finite path between the ancestor and the descendant. Heck (2016) proves that the answer is yes, but this requires some sort of mathematical in-

Fulugonio (2008) also offers an argument in favor of the circularity reading. Her arguments are very similar to Angelelli's[20] as it reiterates that the circularity undermines Frege's project completely. She argues that:

> The only supposition in Kerry's criticisms to the Fregean definition of succession is that Frege's construction, and the definition of succession, in particular, has a defined gnosiological motivation, one that Frege certainly explicit in many opportunities throughout his work. From that, if, as part of his project, Frege intends to clarify the notion of succession, his elucidation is - at least - unsatisfactory if it demands, among other things, that it is known what it attempts to be clarifying. (FULUGONIO, 2008, p.9)[21]

Fulugonio's point is precisely Angelelli's second circularity argument. Frege indeed had a gnosiological program in mind: to show that all arithmetical truths are analytic, not to mention *a priori*, which in turn should prove that all our knowledge on numbers is logically justified. This program includes the so-called reduction of ordinary notions to logical definitions, as the Ancestral. But the crux of Fulugonio's point is the same as before: the hidden assumption that any universally quantified proposition demands a verificational proof, *i.e.,* a proof for each instance. But our conditions for verifying any number of cases – finite or infinite – should not be taken as responsible for the proof of a quantified proposition. Frege did make this point on multiple occasions. He warned in the *Grundlagen* and earlier in the *Begriffsschrift* not to take the *being* true as *taken* to be true. The analyticity of arithmetic should be a consequence of its logical grounds, not the other way around. As far as the Ancestral is concerned, definitions and proofs should not be taken as verificationally motivated.

duction on the number of steps that go from each node of the parent relation. This is worrying because Frege's ancestral supposed to derive a principle of mathematical induction, not to depend on it for its extensional correctness. But Heck also shows that one can define the Ancestral in a way that avoids such circularity and that is still equivalent to (FA).

[20] And in her defense, came earlier than Angelelli's paper.

[21] In the original: "El único supuesto presente en la crítica de Kerry a la definición fregeana de sucesión es que toda la construcción fregeana, y la definición de sucesión en particular, tiene una aspiración gnoseológica definida, aspiración que por cierto Frege explicita en reiteradas oportunidades a lo largo de toda su obra. De modo que, si como parte de su proyecto lo que pretende Frege es elucidar la noción de sucesión, su elucidación es –cuanto menos– insatisfactoria si ella exige, entre otras cosas, que sea ya conocido aquello que se quiere elucidar". The translation is mine.

Furthermore, Fulugonio argues that Russell's defense of Frege's definition of the Ancestral missed one important point in not discussing the problematic property. As she says, "[..] 'to follow $x$' is precisely one of the hereditary properties in which we find ourselves in a vicious circle." (FULUGONIO, 2008, p.8)[22]. But Russell's response goes to the heart of the problem: only by taking an incorrect account on quantification that such property becomes problematic as Kerry's supposed in the first place. In his short response to Angelelli, Heck (2016, p.101) added that "Russell is making an elementary logical *cum* epistemological point: a universal generalization does not have to be derived from the conjunction of its instances, and knowledge of a universal generalization need not rest upon knowledge of its instances".

It should also be noted that Fritz's case is a particular one: a specific application of **(FA)** to a specific individual. For that matter, it's hard to imagine that Fritz could prove $R^*(k,f)$ without a likely empirical (*viz.* intuitive) proof. And without some grounds, he cannot. It seems that **(FA)** is not suitable for helping his case, but again, this was not Frege's interest either. Heck also adds that "the power of Frege's definition shows itself not in particular cases but in results like [...] the *generalizations* that it allows to us to prove" (HECK, 2016, pp.101-102). Take, for example, the transitivity of the Ancestral (Theorem 98). Can we prove that the ordinary ancestral relation is transitive regarding every application in an intuitive way? Can we show that if $n<m$ and $m<p$, then $n<p$? If $n,m,p$ are natural numbers then the number of nodes between $n$ and $p$ is finite. The task might be difficult, but is still possible. But the same cannot be said if $n,m,p$ are extended to the Reals, since the number of nodes between $n$ and $p$ are not finite in this case.

Frege's intentions were with his logicist account of arithmetic, and in this particular case, no "checking" procedure is possible, if numbers are to be regarded as logical objects. Otherwise, propositions about numbers would be empirical ones. As Russell (1996, p.522) completes, "Kerry's argument, therefore, is answered by a correct theory of deduction; and the logical theory of arithmetic is vindicated against its critics". Russell also pointed out that this is

[22] In the original: "[...] que "seguir a $x$" es precisamente una de tales propiedades hereditarias, con lo cual nos encontramos ante un círculo vicioso". The translation is mine.

very akin to Mill's objection to Barbara inferences, but can be likewise answered. He concludes that a

> general proposition can often be established where no means exist of cataloging the terms of the class for which they hold; and even, as we have abundantly seen, general propositions fully stated hold of *all* terms, or, as in the above case, of *all* functions, of which no catalogue can be conceived. (RUSSELL, 1996, p.522)

For which Angelelli's argument can be likewise answered.

The catalog of all functions surely would be problematic for its cardinality, but for matters of both the circularity arguments, any infinite (countable) domain would already offer serious problems. Given that $R^*$ is not restricted to any specific domain of interpretation, applications in domains already known to be infinite would be problematic. Considering the case of the natural numbers, it's clear that they satisfy an infinite number of hereditary properties. For instance, for a given pair of natural numbers $(m, n)$, there exists a natural number $p$ such that $m<p$ and $n<p$. Hence, 'being smaller than $p$' is undoubtedly a property shared by $m$ and $n$ in the predecessor relation. Since there are infinitely many $p$'s, there would be infinitely many such properties.

This would not be a problem if, in principle, the domain of quantification were finite. But we are not even sure how the initial list of properties can be finite in the first place. Because if it is, we can still expand it indefinitely. Assume, for example, that there is such a finite catalog of properties and that **(FA)** is justifiably true just in case there is a way to verify each of the properties in it. This means that $R^*(a,b)$ holds if, and only if, for each $F_{i\leq n}$ that is hereditary, for a finite $n$, it is verified that if all $R$-descendants of $a$ has $F_i$, then $F_i(b)$. This means that $b$ inherits the hereditary properties of $a$. Assume also that all $F_i$'s are decidable, that is, no circularity in Angelelli's terms is derivable from the list[23]. Thus, in theory, if every property is successfully verified, $R^*(a,b)$ is said to be verified as well. But why should we settle for the initial list? If $b$ satisfies $F_i(x)$ then certainly $b$ also satisfies $F_i(x) \vee \varphi$ for any new formula $\varphi$ not originally in the list. Also, if $b$ satisfies any two formulas in

[23] This implies that the property $[x: R^*(a, x)]$ is not in the list.

the list, then it certainly satisfies the conjunction, implication and the biconditional of those same formulas.

From this assumption, it is possible to expand the initial finite list endlessly, starting only from those formulas of properties known to our original list and adding every new formula obtainable from the closure of those formulas under the relevant logical operators. Generally, if $F_i(x)$ and $F_j(x)$ are in the list, for $i, j \leq n$ and $n$ being the number of entries, then, $F_i(x) \vee \varphi$, $F_i(x) \wedge F_j(x)$, $F_i(x) \rightarrow F_j(x)$, $F_i(x) \equiv F_j(x)$ are all in the new list. Clearly, if $b$ satisfies every property of the finite initial list, then $b$ also satisfies all properties obtained in the new list, given that every new entry is also hereditary.

This procedure "generates" new properties. Even if we have only one property in the list, *i.e.,* if $n=1$, the new list will be expanded infinitely, forming every iteration of formulas from $F_1(x)$. The point is that we have no means to reject this infinite expansion if we assume that such a list exists to be verified. Even more, we have no good arguments to deny that this expanded list wasn't the one that we already had in the first place. Thus, if we are to assume a "checking" procedure, which is limited by our finite conditions, we would never check all of them. Once again, any quantification into an infinite domain would be untenable. The Fregean response would be that any proof of $R^*$ should not be restricted to a finite checking whatsoever.

In Fritz's case, we could easily consider that even if he can show every decidable hereditary property that is passed from Karl's descendants to him, the jury could easily form a new list of hereditary properties from the previous one, thus extending Fritz task into infinity. But of course, this rather ordinary procedure is not the only way one can prove that $R^*(a, b)$, or Fritz to prove that he is Karl's descendant[24]. Fritz would easily prove that he has this property if he had grounds to show his heritages. If he is eager to use **(FA)**, these grounds must be logically justified. But it is hard to believe that this is what matters in Fritz's case. If the question was to decide whether for example $P^*(0,3)$ holds, surely there must be a logical way to check it. But Fritz is not a logical object, so there would be no pure logical form to decide whether he is

[24] The same point is made by Heck (2016).

Karl's descendant or not. Angelelli argues that **(FA)** is a reduction to the sense that it's supposed to substitute **(OA)**, to the extent that Frege's intended to provide a more precise language than the ordinary one. Frege's intention, as I see it, certainly was to provide better definitions for ordinary concepts, but *insofar* as they are necessary for mathematical purposes. In this sense, there is no reason to expect from Frege the intentions that Fritz has in the fictitious example that Angelelli provides.

Finally, if we expect Fritz to prove his heritages only in the presence of verification of each instance of the parent relation from him to Karl, Fritz would have problems to show, for example, that if he is the son of someone who is Karl descendant, he is Karl's descendant as well. But, providing these cases, it is clear that he would be, even without a demonstration of the complete path from his father to Karl. If Fritz had those two premises proved, he could prove that he is Karl's descendant with the aid of Theorem 96 of the *Begriffsschrift*:

$$R^*(x, y) \wedge R(y, z) \rightarrow R^*(x, z)$$

Likewise, should one prove that $[x : R^*(a, x)]$ is a hereditary property by looking to every member of the $R$-series? Certainly not. It can be easily proved from the definition alone. And Theorem 97 of the *Begriffsschrift* does exactly that.

## 5. Realism and Impredicativity

As we may conclude, there is no reason to embrace any version of Russell's vicious circle principle within the Fregean realist framework. In fact, the vicious circle principle did not receive full acceptance, as not everyone was in full agreement with the complete ban of such definitions, given that not every impredicative case is harmful or at least problematic. For Ramsey (1931), definitions are just a way of describing or putting into symbols the function or relation defined, since their totality already exists independently from the definition. Ramsey adopted the Wittgensteinian perspective of reading a quantified expression like $\forall F : F(x)$ as the logical sum of all propositions of form $F(x)$. Given, in his terms, "[…] our inability to write propositions of infinite length, which is logically a mere accident" (RAMSEY, 1931, p.41), there is no problem in describing them in terms of the totality in which they may be a member. In the now-famous example, he stated that

> [...] to express it like this [...] is merely to *describe it* in a certain way, by reference to a totality of which it may be itself a member, just as we can refer to a man as the tallest in a group, thus identifying him by means of a totality of which he is himself a member without there being any vicious circle (*ibidem.*)[25].

Thus, Ramsey agrees with the Fregean view that properties are not created, but grasped through definitions. A different stance was proposed by Carnap, since he did not accept the existence of such totality as given, even labeling Ramsey's position as "theological mathematics" (CARNAP, 1964, p. 50). As mentioned earlier, Carnap did recognize that universally quantified expressions need not be verifiable in terms of individual instances:

> If we reject the belief that it is necessary to run through individual cases and rather make it clear to ourselves that the complete verification of a statement about an arbitrary property means nothing more than its logical (more exactly, tautological) validity for an arbitrary property, we will come to the conclusion that impredicative definitions are logically admissible. (CARNAP, 1964, p.50).

He then continued by concluding that, even though it might be difficult to prove some instances of an impredicatively defined property, or even impossible, this difficulty or impossibility may not be a consequence of the impredicativity itself. Carnap's adopted a pragmatic perspective, as he elaborates more in *Logical Syntax of Language*. Accepting or not an impredicative definition is "[...] a question of choosing a form of language — that is, of the establishment of rules of syntax and of the investigation of the consequences of these" (CARNAP, 1937, p.164). In other words, it is just a matter of tracking the consequences and avoiding the problematic ones.

Perhaps more directly in tone with the Fregean solution here proposed was Gödel. He famously argued that impredicative definitions are problematic only by assuming what he calls a "constructivist" approach towards definitions:

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[25] The emphasis is mine.

> [...] it seems that the vicious circle principle [...] applies only if the entities involved are constructed by ourselves. [...] If, however, it is a question of objects that exist independently of our constructions, there is nothing in the least absurd in the existence of totalities containing members which can be described (*i.e.*, uniquely characterized) only by reference to this totality. (GÖDEL, 1944, p.128)

So Gödel took a very similar standpoint to that of Frege, by assuming the objective existence of properties, classes, and the like. He went even further by taking their existence on a par with physical objects: “Classes and concepts may, however, also be conceived as real objects [...] the assumption of such objects is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence” (GÖDEL, 1944, p.128).

Ramsey, Carnap, and Gödel are good examples of the opposite view from that of Russell and Poincaré’s about the vicious circle principle. But Ramsey and Gödel are more representatives of the realist answer that Frege could have given to Kerry’s objections, if he had provided one. Parsons (2014) discuss the positions that Ramsey, Carnap, and Gödel and other key authors such as Hilbert and Bernays. His conclusion goes against the conventional wisdom that the historical defense of impredicative definitions always goes along realist lines. Carnap certainly was one that denied some of the realist presuppositions and still accepted impredicative definitions. Moreover, the acceptance of classical mathematics inevitably forces one to accept impredicativity, and this is not exclusive of realist philosophers. Thus, the acceptance of impredicativity does not imply the adoption of realism. But the other direction might be the case: accepting a realist ontology, at least from the historical point of view, seems to imply the acceptance of impredicativity definitions. At least I know of no historical counter-example.

But in Frege’s case, the implication is surely true, as I’ve tried to argue. The realist stance is an answer to the Fritz case. First, it is clear that Fritz is trying to show his heritage, not to create them, given that he is Karl’s descendant regardless of his argumentation. $R^{*}(k,f)$ holds or not, even if Fritz cannot show that he has all hereditary properties passed from Karl to his descendants. Second, there is an infinite number of such properties, meaning that

Fritz would never be able to show all of them in the first place, assuming such constructivist reading about the Ancestral. The same line of thought follows for any application of the Ancestral, whether about Fritz, Numbers, or any set of objects describing a linear-ordered series.

## 6. Conclusion

The story is that Frege's definition is definitely impredicative, given that $R^*(a, x)$ is in the scope of the second-order quantifier in $R^*$, and Kerry must be praised as perhaps the first one to realize the problem of impredicativity in logic. The problem is also related to the Fregean definition of Natural Number. Recall that $n$ is a natural number just in case $P^+(0, n)$ holds for $n$. But this is also a hereditary property: a successor of a natural number is itself a natural number. But $P^+(0, n)$ holds if $n$ has all hereditary properties of $0$ that all $P$-successors of $0$ have. Hence, to define something as a natural number requires knowing if that something is a natural number, so the alleged circularity tells us. The question is whether this is a problem for Frege. The existence of circularity depends upon some previous philosophical background, and in Frege's realist perspective there seems to be none.

The other place where impredicativity affected Frege was in his full comprehension principle for functions. This, added to Axiom V regarding value-ranges, yields Russell's paradox. It was this problem that motivates Russell in avoiding such impredicative cases. Most of the important theorems regarding the Ancestral are dependent on comprehension axioms as well. Thus, in order to use them, it is necessary to provide some restrictions in order to avoid cases like $\exists F \forall x (F(x) \equiv \neg F(x))$ to be derivable[26]. But even with such restrictions, Frege's Ancestral works as intended. Other problems, and answers, that Frege's Ancestral is subject to are found in Heck (2016). In fact, some of the defense of Frege's definition here taken are found in Heck's paper, precisely on the misconception about quantification assumed by Kerry and Angelelli. What I proposed here, however, was an extended answer that relies more on

[26] In this case, by simply forbidding the formula $F$ to occur free in the right side of the equivalence.

Frege's philosophy. To be historically fair, we can credit the early Russell of the *Principles of Mathematics* for being the first to point out the problem in such objections.

In conclusion, by taking a Fregean stance, and by arguing that Frege's definition of the Ancestral is not circular as both Kerry and Angelelli intended, we can reject the circularity premises in both arguments. In doing so, we are entitled to reject both conclusions, *viz.,* that Frege's definition is not verifiable, and that is not a proper reduction of the ordinary notion. With that, we kept Frege's reduction safe, at least with respect to his philosophical motivations, and showed that he could have been the first realist stance against vicious circle principles.

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